# A simple derivation of Lighthill's heat transfer formula

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## 1. Introduction and summary

In the following it will be shown that a simple argument based on the use of the energy integral equation of the laminar boundary layer permits the derivation of a heat transfer formula valid for non-uniform temperature distribution and non-zero pressure gradients. The formula is then shown to be identical in structure with Lighthill's (1950) well-known results. Lighthill obtained his formula by solving the boundary layer equations in the von Mises form using operational methods. An elegant way to obtain the same results using exact similarity consideration was given by Lagerstrom (not yet published). The derivation given here is probably the most simple-minded one and the method may be useful for other applications as well. Furthermore, it is shown that the approach can be slightly modified to permit application of the formula to flow near separation. The latter result is applied to the Falkner–Skan solution for just separating flows and is found to be in excellent agreement with the exact solutions.

## 2. Laminar flow far from separation

The energy integral equation relating the heat flux $q_w$ at the surface to the enthalpy distribution $h(y)$ through the boundary layer can be written

$$\frac{d}{dx}\int_0^\delta \rho u(h-h_\infty)\,dy = -q_w, \tag{1}$$

where $\delta$, $\rho$, $u$ are boundary layer thickness, density, and $x$-component of the velocity vector, respectively. Heat produced by dissipation has been neglected.

If the Prandtl number $Pr$ is not very small*, $q_w$ is determined by the velocity distribution in the immediate neighbourhood of the wall. This is a well-known result first used by Fage & Falkner (1931). Thus

$$u \doteqdot (\partial u/\partial y)_w y = (\tau_w/\mu_w)y, \tag{2}$$

where $\tau_w$ and $\mu_w$ are shearing stress and viscosity at the wall. We will now express $y$ in formula (2) in terms of $h$ and $q$ in the neighbourhood of the wall and then employ a change of variables in (1) such that $q$ is considered as a function of $h$. This transformation is patterned after Crocco's method of using $\tau(x,u)$ instead of $\tau(x,y)$. We write

$$h = h_w + (\partial h/\partial y)_w y = h_w - (c_p/k)_w q_w y,$$

* The opposite case $Pr \ll 1$ is even easier since one can replace $\rho u$ by $\rho_\infty U$ in (1).

and, on combining this with (2), we have near the wall

$$u = Pr^{-1}\tau_w(h_w - h)/q_w. \tag{3}$$

The heat flux $q$ is given by

$$q = -\frac{k}{c_p}\frac{\partial h}{\partial y}. \tag{4}$$

The integration in (1) is performed at constant $x$ and we can replace $dy$ by $dh$ using (4). Thus (1) can be written

$$\frac{1}{Pr}\frac{d}{dx}\left\{\frac{\tau_w}{q_w}\int_{h_w}^{h_\infty}\frac{(h_w-h)(h-h_\infty)\rho k}{qc_p}\,dh\right\} = q_w,$$

or, if we introduce the similarity variable

$$\eta = \frac{h_w - h}{h_w - h_\infty},$$

where $h_\infty$ stands for the enthalpy at the edge of the layer, we have

$$\frac{1}{Pr^2}\frac{d}{dx}\left\{\frac{\tau_w\mu_w\rho_w}{q_w^2}(h_w-h_\infty)^3\int_0^1\left(\frac{\rho\mu}{\rho_w\mu_w}\right)\frac{\eta(1-\eta)}{q/q_w}\,d\eta\right\} = -q_w. \tag{5}$$

We now make the similarity assumptions that $\rho\mu/(\rho_w\mu_w)$ and $q/q_w$ are functions of $\eta$ only, that is

$$q(x,h) = q_w(x)f\left(\frac{h_w - h}{h_w - h_\infty}\right).$$

While this cannot be exact in all cases, it is probably very closely satisfied for a large range of conditions, a fact which will be borne out by a comparison of the final results with known solutions. If the similarity is granted then the integral in (5) is a constant $\alpha$, say, to be evaluated later. The result (5) becomes a very simple differential equation from which $q_w$ as a function of $\tau_w$ and of $\Delta h = h_w - h_\infty$ is easily obtained. We have from (5)

$$-\frac{1}{q_w}\frac{d}{dx}\left(\frac{\tau_w\mu_w\rho_w\Delta h^3}{q_w^2}\right) = \frac{Pr^2}{\alpha}.$$

Multiplication with $\sqrt{(\tau_w\mu_w\rho_w\Delta h^3)}$ makes the left side a perfect differential, and hence we find that

$$-\frac{\sqrt{(\tau_w\mu_w\rho_w\Delta h^3)}}{q_w} = \left(\frac{3}{2\alpha}\right)^{1/3}Pr^{2/3}\left[\int_0^x\sqrt{(\tau_w\mu_w\rho_w\Delta h^3)}\,dx\right]^{1/3},$$

or

$$-q_w = \left(\frac{2\alpha}{3}\right)^{1/3}Pr^{-2/3}\sqrt{(\tau_w\mu_w\rho_w\Delta h^3)}\left[\int_0^x\sqrt{(\tau_w\mu_w\rho_w\Delta h^3)}\,dx\right]^{-1/3}. \tag{6}$$

The result (6) permits the computation of $q_w$ if $\tau_w(x)$ and $T_w(x)$ or $h_w(x)$ are given.

To determine the constant $\alpha$ we have to choose a reasonable distribution of $q\rho_w\mu_w/q_w\rho\mu = f(\eta)$*. However, since only the cube root of $\alpha$ appears,

* One can also use the known solution for a flat plate with uniform temperature to obtain $\alpha$.

the results are very insensitive to the choice of the function $f(\eta)$. If we base our approach again on Crocco's work for the shearing stress we choose

$$f(\eta) = \sqrt{(1-\eta^2)},$$

a simple function with the proper behaviour at the limits. The constant $\alpha$ then has the value $1 - \frac{1}{4}\pi \doteqdot 0{\cdot}215$. To compare (6) with Lighthill's (1950) formula it is easiest first to find $q_w(x)$ from (6) if $\Delta h$ vanishes in the range $(0, \xi)$ and then jumps to a constant value $\Delta h_0$ for greater values of $x$. In this way (6) can be written in the form of a Stieltjes integral like Lighthill's formula, and the comparison is as follows.

Lighthill's result (1950, equation (29), with $\rho\mu$ assumed constant) is

$$-q_w(x) = \frac{9^{-1/3}}{\frac{1}{3}!}\, k\left(\frac{\rho Pr}{\mu^2}\right)^{1/3} \sqrt{\{\tau_w(x)\}} \int_0^x \left[\int_\xi^x \sqrt{\{\tau_w(\xi)\}}\, d\xi\right]^{-1/3} dT_0(\xi).$$

Equation (6) transforms into exactly the same form but with a numerical constant $(\frac{2}{3}\alpha)^{1/3}$ instead of $9^{-1/3}/\frac{1}{3}!$. With the value $\alpha = 0{\cdot}215$ determined above we obtain $0{\cdot}524$ for the constant compared with Lighthill's value of $0{\cdot}539$. The agreement is thus better than 3%. Even with the worst possible choice of $f(\eta)$, namely $f(\eta) = 1$, our constant would be reduced to only $0{\cdot}490$.

## 3. Laminar flow near separation

It is now interesting to modify the approach to handle the flow near the separation point where $\tau_w \to 0$. Here (2) obviously fails to represent $u$ and the approximation becomes bad. However, it is an easy matter to try an approximation which represents $u$ near a separation point. Here,

$$u \doteqdot \tfrac{1}{2}y^2(\partial^2 u/\partial y^2)_w \tag{7}$$

but

$$\mu_w(\partial^2 u/\partial y^2)_w = dp/dx. \tag{8}$$

Hence

$$u = (y^2/2\mu_w)\, dp/dx,$$

and, proceeding as before, we obtain

$$-q_w = (\tfrac{3}{8}\beta)^{1/4} Pr^{-3/4}\left(u_w^2 \rho_w \frac{dp}{dx}\Delta h^4\right)^{1/3}\left[\int_0^x \left(u_w^2\rho_w \frac{dp}{dx}\Delta h^4\right)^{1/3} dx\right]^{-1/4}, \tag{9}$$

where $\beta$ stands for the integral

$$\int_0^1 \frac{\rho\mu}{\rho_w\mu_w}\,\frac{\eta^2(1-\eta)}{q/q_w}\, d\eta.$$

The result (9) can be compared with the Falkner–Skan solution for just-separating flow. Lighthill gives a table showing the ratio (Nusselt number)/$\sqrt{}$(Reynolds number) for Falkner Skan flow and $Pr = 0{\cdot}7$. The value given for this ratio for separating Falkner–Skan flow is

$$Nu/\sqrt{(Re_x)} = 0{\cdot}438.$$

The value obtained from (9), again using $f(\eta) = \sqrt{(1-\eta^2)}$ in the evaluation of $\beta$, comes out to be

$$Nu/\sqrt{(Re_x)} = 0{\cdot}448,$$

and agrees also within better than 3%. In addition (9) shows explicitly the dependence of heat transfer on the Prandtl number near separation.

## 4. Extension to high Mach number flow

The exact form of equation (1), valid for arbitrary Mach number, is

$$\frac{d}{dx}\int_0^\delta [h+\tfrac{1}{2}u^2-(h_\infty+\tfrac{1}{2}U^2)]\rho u\, dy = -q_w. \tag{10}$$

Introducing the recovery enthalpy $h_r$, we may write the integral in the form

$$\int_0^\delta [(h-h_r)+(h_r-h_\infty)-\tfrac{1}{2}(U^2-u^2)]\rho u\, dy.$$

It is often possible to neglect the terms

$$h_r-h_\infty-\tfrac{1}{2}(U^2-u^2)$$

and extend the derivations to the high Mach number case simply by replacing $h_\infty$ by $h_r$. It is also possible of course to use the full equation (10), but then the differential equation becomes more difficult and the essential simplicity of the approach is lost.

## References

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